Conditional Entanglement

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Based on the ideas of *quantum extension* and *quantum conditioning*, we propose a generic approach to construct a new kind of entanglement measures called *conditional entanglement*. The new measures, built from the known entanglement measures, are convex, automatically *super-additive*, and even smaller than the regularized versions of the generating measures. More importantly, new measures can also be built directly from measures of correlations, enabling us to introduce an *additive* measure and generalize it to a multipartite entanglement measure.

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Entanglement, as a key resource and ingredient in quantum information and computation as well as communication, plays a crucial role in quantum information theory. It is necessary to quantify entanglement from different standpoints. A number of entanglement measures have been formulated, and their properties have been explored extensively (see, e.g., Ref.[1, 2] and references therein). Nevertheless, little is known on how to systematically introduce new entanglement measures. It is likely accepted that an appropriate entanglement measure is necessarily non-increasing under local operations and classical communication (LOCC), while this requirement makes the definition of entanglement measure notoriously difficult and challenging. So far, most of existing methods to construct entanglement measures are based on the "convex roof" [3] and the concept "distance" [4] — the distance from the entangled state to its closest separable state. The well-known entanglement of formation E_f [3] is established for a mixed state ρ_{AB} of a bipartite AB-system via the technique of convex roof. On the other hand, the relative entropy of entanglement E_r was based on a concept of "distance" [4], and squashed entanglement E_{sq} was built from conditional quantum mutual entropy [5]—a quantum analog to intrinsic information [6] known from classical cryptography, as well as the logarithmic negativity E_N was suggested [7, 8] on the basis of the well-known separability criterion —partial transposition [9]. Among the known measures, additivity holds for E_{sq} and E_N and is conjectured to hold for E_f, but E_r is nonadditive [10]. E_N is computable for a generic mixed state, while it does not reduce to the von Neumann entropy of subsystem for pure states. E_r can be generalized to a measure for multipartite states, but still it is nonadditive. Very recently, E_{sq} was extended to multipartite cases [11].

In this paper, we introduce a generic approach to construct a kind of entanglement measures, which is defined in analogy to the conditional entropy [12] and thus referred to as *conditional entanglement*. The key ideas are quantum extension and quantum conditioning [12]. New entanglement measures can be built from old ones and the order between them is known. Of particular importance, conditional entanglement can be formulated by quantum conditioning of functions that describe correlations rather than entanglement. Taking the quantum mutual information as as exemplary measure of correlations, we show that a new entanglement measure can be established by quantum conditioning. Remarkably, it is additive and can straightforwardly be generalized to multipartite states for two different choices of multipartite mutual information.

Definition 1 Let ρ_{AB} be a mixed state on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. A conditional entanglement of ρ_{AB} is defined as

$$CE(\rho_{AB}) = \inf\{E(\rho_{AA':BB'}) - E(\rho_{A':B'})\}, \qquad (1)$$

where the infimum is taken over all extensions of ρ_{AB}, i.e., over all states satisfying the equation $\mathrm{Tr}_{A'B'}\rho_{AA'BB'} = \rho_{AB}$, and $E(\cdot)$ is an entanglement measure. Note that the above definition is similar to that of conditional entropy $S(A|B) = S(AB) - S(B)$ with $S(\rho)$ as the von Neumann entropy $S(\rho) = -\mathrm{Tr}\rho \log \rho$.

To show that conditional entanglement is a good entanglement measure, we now elaborate that it does satisfy two essential axioms that an entanglement measure should obey [1].

1. Entanglement does not increase under local operations and classical communication (LOCC) i. e. $E(\Lambda(\rho)) \leq E(\rho)$*, for any LOCC operation* Λ. The reason that CE inherits the monotonicity of E is straightforward,

$$E(\Lambda^{AB}(\rho_{AA':BB'})) - E(\rho_{A':B'}) \leq E(\rho_{AA':BB'}) - E(\rho_{A':B'}).$$

2. Entanglement is not negative and is zero for separable states. The inequality $CE(\rho_{AB}) \geq 0$ comes from the fact that any entanglement measure is non-increasing by tracing subsystems, while the equality $CE = 0$ for separable states lies in that separable extensions can be found for separable states.

The monotonicity under LOCC implies that entanglement remains invariant under local unitary transformations. This comes from the fact local unitary transformations are reversible LOCC. The convexity of entanglement used to be considered as a mandatory ingredient of the mathematical formulation of monotonicity [1, 13]. Now the convexity is merely a convenient mathematical property. Also there is a common agreement that the strong monotonicity—monotonicity *on average* under LOCC is unnecessary but useful [1, 13]. Many known existing entanglement measures are convex and satisfy the strong monotonicity. We will show that CE naturally inherits these properties.

For convex E, convexity of CE can be obtained by noticing that for any extension states $\rho_{AA'BB'}$ and $\sigma_{AA'BB'}$, a new extension state can be constructed as $\tau_{AA'E:BB'} = \lambda\rho_{AA'BB'} \otimes (|0\rangle\langle 0|)_E + (1-\lambda)\sigma_{AA'BB'} \otimes (|1\rangle\langle 1|)_E$, and therefore

$$\begin{aligned} & E(\tau_{AA'E:BB'}) - E(\tau_{A'E:B'}) \\ = \;& \lambda[E(\rho_{AA':BB'}) - E(\rho_{A':B'})] \\ + \;& (1-\lambda)[E(\sigma_{AA':BB'}) - E(\sigma_{A':B'})]. \end{aligned} \tag{2}$$

Now, let us show that $CE(\cdot)$ satisfies the monotonicity on average under LOCC if the convex $E(\cdot)$ does. It is sufficient to prove that CE is non-increasing under measurement on one party. For any extension $\rho_{AA'BB'}$, a measurement on party A reduces the extension state to an ensemble $\{p_k, \tilde{\rho}^k_{AA'BB'}\}$.

$$\begin{aligned} & E(\rho_{AA'BB'}) - E(\rho_{A'B'}) \\ \geq \;& \sum_k p_k E(\tilde{\rho}^k_{AA'BB'}) - E(\rho_{A'B'}) \\ = \;& \sum_k p_k E(\tilde{\rho}^k_{AA'BB'}) - \sum_k p_k E(\tilde{\rho}^k_{A'B'}) \\ + \;& \sum_k p_k E(\tilde{\rho}^k_{A'B'}) - E(\rho_{A'B'}) \\ \geq \;& \sum_k p_k [E(\tilde{\rho}^k_{AA'BB'}) - E(\tilde{\rho}^k_{A'B'})]. \end{aligned} \tag{3}$$

The first inequality comes from the fact that E is non-increasing on average under local measurement, while the second one is due to the convexity of E. As a result, we have $CE(\rho_{AB}) \geq \sum_k p_k CE(\tilde{\rho}^k_{AB})$.

Remarkably, while most of the known entanglement measures are sub-additive, CE is super-additive.

Proposition 1 $CE(\rho \otimes \sigma) \geq CE(\rho) + CE(\sigma)$.

Proof For any extension state $\tau_{A_1A_2A':B_1B_2B'}$ of $\rho_{A_1B_1} \otimes \sigma_{A_2B_2}$,

$$\begin{aligned} & E(\tau_{A_1A_2A':B_1B_2B'}) - E(\tau_{A':B'}) \\ = \;& E(\tau_{A_1A_2A':B_1B_2B'}) - E(\tau_{A_2A':B_2B'}) \\ + \;& E(\tau_{A_2A':B_2B'}) - E(\tau_{A':B'}) \\ \geq \;& CE(\rho) + CE(\sigma). \end{aligned} \tag{4}$$

Some entanglement measures are upper bounds for distillable entanglement. Their so-called *regularizations* provide stronger bounds. Here CE is even smaller than the regularized entanglement measure:

$$CE(\rho) \leq E^\infty(\rho) \quad \text{for all states } \rho, \tag{5}$$

where $E^\infty(\rho) = \lim_{n\to\infty} E(\rho^{\otimes n})/n$ is the regularized version of the generating entanglement measure E. Indeed, it is explicit that $CE(\rho) \leq E(\rho \otimes |00\rangle\langle 00|) - E(|00\rangle\langle 00|) = E(\rho)$. From the super-additivity of CE, we know $nCE(\rho) \leq CE(\rho^{\otimes n}) \leq E(\rho^{\otimes n})$, which leads to (5).

One also finds that $E_f(\rho_{AB:CD}) - E_f(\rho_{C:D}) \geq G(\rho_{A:B})$, where $G(\rho_{A:B}) > 0$ iff ρ_{AB} is entangled [14]. We then get $G(\rho_{AB}) \leq CE_f(\rho_{AB}) \leq E_c$, where $E_c = E_f^\infty$ is so-called entanglement cost [15]. Thus for any entangled state, $CE_f > 0$. It is an open question, whether CE_r is nonzero for entangled states.

Now let us pass to constructing entanglement measures by conditioning *correlation measures* [25]. Most intriguingly, we illustrate below that a new additive measure can indeed be constructed based on quantum conditioning and can be generalized to multipartite states.

For a function f quantifying correlations we have two candidates for its conditioned version

$$C^s_f(\rho_{AB}) = \inf[f(\rho_{AA':BB'}) - f(\rho_{A':B'})], \tag{6a}$$

$$C^a_f(\rho_{AB}) = \inf[f(\rho_{A:BE}) - f(\rho_{A:E})], \tag{6b}$$

where infimum is taken over all extensions $\rho_{AA'BB'}$ (ρ_{ABE}) ρ_{AB}. $C^s_f(\cdot)$ is the symmetric conditioned version of f while $C^a_f(\cdot)$ the asymmetric one.

Taking f to be quantum mutual information $I(X:Y) = S(X) + S(Y) - S(XY)$, we obtain *conditional entanglement of mutual information* given by C^s_I. We add a factor $1/2$ and will denote it by C_I. Explicitly

$$C_I(\rho_{AB}) = \inf \frac{1}{2}\{I(AA':BB') - I(A':B')\}, \tag{7}$$

where the infimum is taken over all the extension states $\rho_{AA'BB'}$ of ρ_{AB}. Now we justify that C_I is an appropriate entanglement measure.

1. We prove that C_I satisfies the strong monotonicity. From a symmetry consideration, it is sufficient to prove that C_I is non-increasing under a measurement on subsystem A, namely, $C_I(\rho_{AB}) \geq \sum_k p_k C_I(\tilde{\rho}^k_{AB})$, where $\tilde{\rho}^k_{AB} = A_k\rho_{AB}A_k^\dagger/p_i$, $p_i = tr A_k\rho_{AB}A_k^\dagger$, and $\sum_k A_k^\dagger A_k = I_A$. Another way to describe the measurement process is as following. First, one attaches two ancillary systems A_0 and A_1 in states $|0\rangle_{A_0}$ and $|0\rangle_{A_1}$ to system AB. Secondly, a unitary operation $U_{AA_0A_1}$ on AA_0A_1 is performed. Thirdly, the system A_1 is traced out to get the state as $\tilde{\rho}_{A_0AB} = \sum_k A_k\rho_{AB}A_k^\dagger \otimes (|k\rangle\langle k|)_{A_0}$. Now for any extension state $\rho_{AA'BB'}$, we get the state after the measurement on A, $\tilde{\rho}_{A_0AA'BB'} = \sum_k A_k\rho_{AA'BB'}A_k^\dagger \otimes$

$(|k\rangle\langle k|)_{A_0} = \sum_k p_k \tilde{\rho}^k_{AA'BB'} \otimes (|k\rangle\langle k|)_{A_0}$. Most crucially, we have

$$\begin{aligned}
& I(\rho_{AA':BB'}) - I(\rho_{A':B'}) \\
= \;& I(0_{A_0A_1} \otimes \rho_{AA':BB'}) - I(\rho_{A':B'}) && \text{(8a)}\\
= \;& I(U_{A_0A_1A}(0_{A_0A_1} \otimes \rho_{AA':BB'})) - I(\rho_{A':B'}) && \text{(8b)}\\
\geq \;& I(\tilde{\rho}_{A_0AA':BB'}) - I(\tilde{\rho}_{A':B'}) && \text{(8c)}\\
= \;& \sum_k p_k[I(\tilde{\rho}^k_{AA':BB'}) - I(\tilde{\rho}^k_{A':B'})] \\
+ \;& \sum_k p_k I(\tilde{\rho}^k_{A':B'}) - I(\tilde{\rho}_{A':B'}) \\
+ \;& S(\tilde{\rho}_{BB'}) - \sum_k p_k S(\tilde{\rho}^k_{BB'}) \\
= \;& \sum_k p_k[I(\tilde{\rho}^k_{AA':BB'}) - I(\tilde{\rho}^k_{A':B'})] \\
+ \;& \chi(BB') + \chi(A'B') - \chi(A') - \chi(B') \\
\geq \;& \sum_k p_k[I(\tilde{\rho}^k_{AA':BB'}) - I(\tilde{\rho}^k_{A':B'})] && \text{(8d)}
\end{aligned}$$

where $\chi(\rho) = S(\rho) - \sum_k p_k S(\rho^k)$ is the Holevo quantity of the ensemble $\{p_k, \rho^k\}$. The equality of (8b) comes from that quantum mutual information is invariant under local unitary operation, while the inequalities of (8c) and (8d) stem from, respectively, the facts that quantum mutual information and the Holevo quantity are non-increasing by tracing subsystem. Consequently, we prove that C_I is non-increasing on average under LOCC operation.

2. $C_I \geq 0$ comes from the fact that the quantum mutual information is non-increasing under tracing subsystems of both sides. For a separable state ρ_{AB}, it can always be decomposed into a separable form: $\rho_{AB} = \sum_{i,j} p_{ij} \phi^i_A \otimes \phi^j_B$. An extension state may be chosen to be $\rho_{AA'BB'} = \sum_{i,j} p_{ij} \phi^i_A \otimes (|i\rangle\langle i|)_{A'} \otimes \phi^j_B \otimes (|j\rangle\langle j|)_{B'}$. It is obvious that $I(AA' : BB') = I(A' : B')$, and thus $C_I = 0$ for separable states.

Continuity. The conditional entanglement of quantum mutual information is asymptotically continuous, i.e. if $|\rho_{AB} - \sigma_{AB}| \leq \epsilon$, then $|C_I(\rho) - C_I(\sigma)| \leq K\epsilon \log d + O(\epsilon)$, where $|\cdot|$ is the trace norm for matrix, K is a constant, $d = \dim \mathcal{H}_{AB}$, and $O(\epsilon)$ is any function that depends only on ϵ (in particular, it does not depend on dimension) and satisfies $\lim_{\epsilon\to 0} O(\epsilon) = 0$.

The proof of the asymptotic continuity is similar to that for the squashed entanglement and is presented in the Appendix.

Convexity. C_I is convex, i.e., $C_I(\lambda\rho + (1-\lambda)\sigma) \leq \lambda C_I(\rho) + (1-\lambda)C_I(\sigma)$ for $0 \leq \lambda \leq 1$.

Proof For any extension states $\rho_{AA'BB'}$ and $\sigma_{AA'BB'}$, we consider the extension state $\tau_{AA'A''BB'B''} = \lambda\rho_{AA'BB'} \otimes (|0\rangle\langle 0|)_{A''} \otimes (|0\rangle\langle 0|)_{B''} + (1-\lambda)\sigma_{AA'BB'} \otimes (|1\rangle\langle 1|)_{A''} \otimes (|1\rangle\langle 1|)_{B''}$, and have $I(\tau_{AA'A'':BB'B''}) - I(\tau_{A'A'':B'B''}) = \lambda[I(\rho_{AA':BB'}) - I(\rho_{A':B'})] + (1-\lambda)[I(\sigma_{AA':BB'}) - I(\sigma_{A':B'})]$. This implies C_I is convex.

An immediate corollary of convexity is that $C_I \leq E_f$ and furthermore $C_I \leq E_c$ due to the following additivity.

Proposition 2 $C_I(\rho_{AB} \otimes \sigma_{CD}) = C_I(\rho_{AB}) + C_I(\sigma_{CD})$.

Proof On the one hand, for any extension states $\rho_{AA'BB'}$ and $\sigma_{CC'DD'}$, $\rho_{AA'BB'} \otimes \sigma_{CC'DD'}$ is an extension state of $\rho_{AB} \otimes \sigma_{CD}$.

$$\begin{aligned}
& I(AA'CC' : BB'DD') - I(A'C' : B'D') \\
= \;& I(AA' : BB') - I(A' : B') \\
+ \;& I(CC' : DD') - I(C' : D'). && \text{(9)}
\end{aligned}$$

So $C_I(\rho_{AB} \otimes \sigma_{CD}) \leq C_I(\rho_{AB}) + C_I(\sigma_{CD})$ holds.

On the other hand, for extension states $\tau_{ACE':BDF'}$ of $\rho_{AB} \otimes \sigma_{CD}$, $\tau_{ACE':BDF'}$ is an extension state of ρ_{AB} and $\tau_{CE':DF'}$ is an extension state of σ_{CD}. Therefore we have

$$\begin{aligned}
& I(ACE' : BDF') - I(E' : F') \\
= \;& I(ACE' : BDF') - I(CE' : DF') \\
+ \;& I(CE' : DF') - I(E' : F'). && \text{(10)}
\end{aligned}$$

This means that $C_I(\rho_{AB} \otimes \sigma_{CD}) \geq C_I(\rho_{AB}) + C_I(\sigma_{CD})$. So we have finally the additivity equality.

It is quite remarkable that the property of additivity is rather easy to prove for conditional entanglement while it is extremely tough for other candidates. The reason lies in that the conditional entanglement is naturally supper-additive while others are usually sub-additive. Also the proof for the conditional entanglement shares a similarity with that of squashed entanglement. As a matter of fact, squashed entanglement can be constructed in the same spirit: it is based on asymmetric conditioning of mutual information

$$E_{sq}(\rho_{AB}) = \frac{1}{2}\inf\{I(A : BE) - I(A : E)\} \equiv \frac{1}{2}C^a_I(\rho_{AB}), \tag{11}$$

where the infimum is taken all extensions ρ_{ABE} of ρ_{AB}. It is notable that $I(A : BE) - I(A : E) = I(AE : B) - I(E : B)$ is symmetric w.r.t. systems AB though each term in the formula is asymmetric w.r.t. both parties. This gives the possibility to build symmetric entanglement measures by asymmetric conditioning.

In [11], we call the squashed entanglement *q-squashed entanglement* E^q_{sq} because the extension is generic and the system E is required to be quantum memory. If we restrict E to classical memory, another proper entanglement measure—*c-squashed entanglement* E^c_{sq} can be obtained [11]. Here we show the order relation among these three measures.

Proposition 3 $E^q_{sq} \leq C_I \leq E^c_{sq}$.

Proof. $E^q_{sq} \leq C_I$ comes from the chain rule for quantum mutual information.

$$\begin{aligned}
& I(AA' : BB') - I(A' : B') \\
= \;& I(A' : BB') + I(A : BB'|A') - I(A' : B') \\
= \;& I(A' : B|B') + I(A : B'|A') + I(A : B|A'B') \\
\geq \;& I(A : B|A'B'). && \text{(12)}
\end{aligned}$$

The proof of $C_I \le E^c_{sq}$ is as follows. For the optimal extension for E^c_{sq}, $\rho_{ABE} = \sum p_i \rho^i_{AB} \otimes (|i\rangle\langle i|)_E$, we have a four-partite state $\rho_{AA'BB'} = \sum p_i \rho^i_{AB} \otimes (|i\rangle\langle i|)_{A'} \otimes (|i\rangle\langle i|)_{B'}$, then $I(AA' : BB') - I(A' : B') = \sum_i p_i I(\rho^i_{AB})$.

Once we have the order of the above three measures, we can easily demonstrate that C_I is lockable i.e. that one can decrease it about arbitrary value while removing a single qubit [17, 18, 19]. The example is the *flower state* $\rho_{A_1A_2B_1B_2}$ [18, 19] defined by its purification:

$$|\Psi\rangle^{A_1A_2B_1B_2C} = \frac{1}{\sqrt{2d}} \sum_{\substack{i=1\dots d\\ j=0,1}} |i\rangle^{A_1}|j\rangle^{A_2}|i\rangle^{B_1}|j\rangle^{B_2}U_j|i\rangle^C,$$

where $U_0 = I$ and U_1 is the Fourier transformation of the computational basis $\{|i\rangle\}$. It is shown in [19] that $E^q_{sq} = 1 + \frac{1}{2}\log d$ and furthermore the optimal extension is trivial (the state itself) that is also one of extensions for C_I and E^c_{sq}. If A_2 is lost, then $\rho_{A1:B_1B_2}$ is separable. From *Prop 3*, we immediately obtain that C_I and E^c_{sq} are lockable.

It should be emphasized that one is unable to prove E^c_{sq} to be additive at present, but the three measures are so similar that they are probably the same. If it is the case, we would have a really graceful result that the optimal extension is always the classical one. Moreover, it would give us a strong hint for the additivity of entanglement of formation that relates to many other important problems [20]. Presumably C_I may play a role as a bridge.

Among existing bipartite entanglement measures [1, 2], only the relative entropy of entanglement and the squashed entanglement can be extended to multipartite cases. Attractively, there exist two versions of multipartite quantum mutual information [21]. All conclusions for the bipartite case can be similarly deduced.

We then obtain two multipartite versions of C_I:

$$C_I = \inf\{I_n(A_1A'_1 : \cdots : A_nA'_n) - I_n(A'_1 : \cdots : A'_n)\},$$
$$C_S = \inf\{S_n(A_1A'_1 : \cdots : A_nA'_n) - S_n(A'_1 : \cdots : A'_n)\},$$

where two candidates for multipartite mutual information are defined as $I_n = \sum_i S(A_i) - S(A_1 \cdots A_n)$, and $S_n = \sum_i S(A_1 \cdots A_{i-1}A_{i+1} \cdots A_n) - (n-1)S(A_1 \cdots A_n)$.

Proposition 4 The conditional entanglement for multipartite mutual information is additive.

$$C_I(\rho_{A_1\cdots A_n} \otimes \sigma_{B_1\cdots B_n}) = C_I(\rho_{A_1\cdots A_n}) + C_I(\sigma_{B_1\cdots B_n}),$$
$$C_S(\rho_{A_1\cdots A_n} \otimes \sigma_{B_1\cdots B_n}) = C_S(\rho_{A_1\cdots A_n}) + C_S(\sigma_{B_1\cdots B_n}).$$

In summary, we have developed a generic approach to construct new entanglement measures based on quantum conditioning. The new measures can not only be obtained from the known measures but also be generated from measures of correlations. In particular, a new additive measure is constructed and generalized to multipartite entanglement. Moreover, the known additive measure—squashed entanglement is shown to come from the asymmetric conditioning. We conjecture that the measures built from quantum conditioning are additive, which means that quantum conditioning leads to additive entanglement. Conditional entanglement measures from other candidates and further properties will be addressed elsewhere.

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APPENDIX

Proof of the asymptotic continuity of C_I

The proof is similar to the continuity of the squashed entanglement [5] that is based on a basic result in [22] asserting that for any two states ρ_{AB} and σ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, if $|\rho_{AB} - \sigma_{AB}| = \epsilon$, then

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq 4\epsilon \log d_A + 2H(\epsilon), \qquad (13)$$

where d_A is the dimension of $\mathcal{H}_A$ and $H(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon)\log(1-\epsilon)$. Note that the righthand of Eq (13) does not explicitly depend on the dimension of $\mathcal{H}_B$. Iteratively using the relations between fidelity and trace norm [23], if $|\rho_{AB} - \sigma_{AB}| \leq \epsilon$, then the fidelity $F(\rho_{AB}, \sigma_{AB}) \geq 1 - \epsilon$, then there exist purifications Φ_{ABC} and Ψ_{ABC} of ρ_{AB} and σ_{AB} respectively such that $F(\Phi_{ABC}, \Psi_{ABC}) \geq 1-\epsilon$, and then $|\Phi_{ABC} - \Psi_{ABC}| \leq 2\sqrt{\epsilon}$. For any quantum operation $\mathcal{E}$ acting on C into $A'B'$, it creates the extensions $\rho_{AA'BB'}$ and $\sigma_{AA'BB'}$ of ρ_{AB} and σ_{AB} satisfying $|\rho_{AA'BB'} - \sigma_{AA'BB'}| \leq 2\sqrt{\epsilon}$. Notice that $I(AA' : BB') - I(A' : B') = S(A|A') + S(B|B') - S(AB|A'B')$, we get

$$\begin{aligned}
& |[I(AA' : BB')_\rho - I(A' : B')_\rho] \\
- & \ [I(AA' : BB')_\sigma - I(A' : B')_\sigma]| \\
= & \ |[S(A|A')_\rho - S(A|A')_\sigma] + [S(B|B')_\rho - S(B|B')_\sigma] \\
- & \ [S(AB|A'B')_\rho - S(AB|A'B')_\sigma] \\
\leq & \ |S(A|A')_\rho - S(A|A')_\sigma + |S(B|B')_\rho - S(B|B')_\sigma| \\
+ & \ |S(AB|A'B')_\rho - S(AB|A'B')_\sigma| \\
\leq & \ 16\sqrt{\epsilon}\log(d_A d_B) + 6H(2\sqrt{\epsilon}) = \epsilon' \qquad (14)
\end{aligned}$$

For a sequence of operation $\mathcal{E}_i$ that creates a sequence of extensions such that $I(AA' : BB')_\rho - I(A' : B')_\rho \to E_I(\rho_{AB})$, we have $|C_I(\rho_{AB}) - [I(AA' : BB')_\sigma - I(A' : B')_\sigma]| \leq \epsilon'$, then $C_I(\sigma_{AB}) \leq I(AA' : BB')_\sigma - I(A' : B')_\sigma \leq C_I(\rho_{AB}) + \epsilon'$. Similarly $C_I(\rho_{AB}) \leq C_I(\sigma_{AB}) + \epsilon'$, so $|C_I(\rho_{AB}) - C_I(\sigma_{AB})| \leq \epsilon'$.

Notice that we have $\sqrt{\epsilon}$ instead of ϵ, but it does not change the essence of condition referring asymptotic continuity [24].

Definition of E^c_{sq} [11] The c-squashed entanglement E^c_{sq} is defined as

$$E^c_{sq}(\rho_{AB}) = \inf \frac{1}{2} I(A : B|E) \qquad (15)$$

where infimum is taken over the extension states of the form $\sum p_i \rho^i_{AB} \otimes (|i\rangle\langle i|)_E$.

In deed, it is equivalent to the mixed convex roof of the quantum mutual information, i.e.

$$E^c_{sq}(\rho_{AB}) = \min \frac{1}{2} \sum_i p_i I(\rho^i_{AB}), \qquad (16)$$

where $\rho_{AB} = \sum_i p_i \rho^i_{AB}$.